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PARETO SURFACES OF COMPLEXITY 1

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13. ABSTRACT

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page 9, line 13: Since each h_i is nondecreasing and convex, this...

page 10, line 1: $P = \bigcup\limits_{x \in (0,1]} \operatorname{conv}\{(x,0,\sqrt{1-x^2}),\ (0,1,1)\}$

page 14, last line: $\frac{1}{2}x_j^2 = \frac{1}{2}(1-x_k^2)$,

page 15, line 3 from bottom: $V^S(0) = T^{|S|-1} - R_+^{|S|}$

page 16, line 7: $V^S(0) = T^{|S|-1} - R_+^{|S|}$

line 13: ...Theorem 3.6 by more general collections.

page 17, line 9: $V = \operatorname{conv}(A^{23} \cup A^{13} \cup A^{12}) - R_+^3$

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PARETO SURFACES OF COMPEXITY 1

by

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ABSTRACT

Pareto surfaces and attainable sets of complexity 1 (i.e., those having a 1-commodity representation and no 0-commodity representation) are treated. An implicit characterization of these sets is given, and various of their properties are derived. In particular, Pareto surfaces of complexity at most 1 are always closed sets.

§1. INTRODUCTION

In [3], the authors define and characterize the attainable sets and Pareto surfaces for systems of n concave, continuous real functions defined on the unit m-cube I^m. In that work, the notion of complexity of an attainable set (and its associated Pareto surface) is defined and briefly discussed. It is the purpose of this paper to study those attainable sets and Pareto surfaces having complexity equal to 1. We shall give an implicit characterization of complexity 1 attainable sets and derive some of its consequences. One of these is that complexity 1 Pareto surfaces are always closed.

Let $I^m = [0,1]^m$ be the unit m-cube, where m is a positive integer. We take $I^0 = \{0\}$. For $m > 0$, let $e^m = (1,\ldots,1) \in I^m$, and take $e^0 = 0$. Let $m \geq 0$, and suppose $u_i : I^m \to R$ are concave, continuous functions for $i = 1,\ldots,n$

<u>Definition 1.1</u>: The <u>attainable set</u> for $u_1,\ldots u_n$ is the set

(1.1.1) $$V = \{x \in R^n | x_i \leq u_i(y^i);\ y^i \in I^m,\ \sum_{i=1}^{n} y^i = e^m\}.$$

The <u>Pareto surface</u> for $u_1,\ldots,u_n$ is the set

(1.1.2) $$P = \{x \in V | y \in V,\ y \geq x \Rightarrow y = x\}.$$

In [3], attainable sets in R^n are characterized as all sets of the form $C - R^n_+$, where $C \subset R^n$ is compact and convex, $R^n_+ = [0,\infty)^n$, and the minus denotes algebraic subtraction of sets. By definition, the Pareto surface associated with an attainable set V is the set of all maximal elements of V with respect to the normal partial order on R^n. A set P is a Pareto surface if and only if P is bounded, P contains no two distinct comparable elements, and $P - R^n_+$ is closed and convex (see [3]). Pareto surfaces need not be closed sets.

Let V be an attainable set in R^n. If $u_1,\ldots,u_n$ are continuous, concave functions on I^m, for some m, such that V is given by (1.1.1) for these u_i's, then the u_i's will be called a representation for V over I^m. We define the complexity of V (denoted $\text{com } V$) to be the least $m \geq 0$ such that there exists a representation for V over I^m. The complexity of a Pareto surface P will be defined to be the complexity of the associated attainable set $V = P - R^n_+$.

It is easy to see that $\text{com } V = 0$ if and only if $V = \{x\} - R^n_+$ for some $x \in R^n$ (and this implies a unique complexity 0 representation: $u_i \equiv x_i$ on I^0). Such a set will be called a corner (or a corner on x).

A corollary of the characterization of [3] is the fact that $\text{com } V \leq n(n-1)$ for any attainable set in R^n. There it is conjectured that this number can be reduced to $n-1$. This conjecture is easily verified for $n = 1$ and 2, but remains unsettled already for $n = 3$. Our approach here will be somewhat different: we will consider those attainable sets in R^n having complexity 1, where n is unspecified.

The idea of considering attainable sets and Pareto surfaces arises in n-person game theory and mathematical economics. For work related to this subject, see [2], [4] and [5].

§2. A CHARACTERIZATION OF COMPLEXITY 1 ATTAINABLE SETS

Throughout the remainder of the paper $V \subset R^n$ will be an attainable set. Denote $N = \{1,\ldots,n\}$. The following lemma allows us to extend concave nondecreasing functions without destroying concavity.

Lemma 2.1: Suppose $a < b < c$ and $f: [a,b] \to R$ is concave, continuous and nondecreasing. Define $g: [a,c] \to R$ by $g|[a,b] = f$ and $g(x) = f(b)$ for $x \in [b,c]$. Then g is concave, continuous and nondecreasing.

Proof: The function g is clearly continuous and nondecreasing. It is concave on $[a,b]$ since f is, and it is concave on $[b,c]$ since it is constant. Thus to prove concavity of g on $[a,c]$ it suffices to take $x \in [a,b)$, $y \in (b,c]$, $\alpha \in (0,1)$ and show $g(\alpha x + (1-\alpha)y) \geq \alpha g(x) + (1-\alpha)g(y)$. Put $z = \alpha x + (1-\alpha)y$. If $z \geq b$ then since g is nondecreasing we have $g(z) \geq \max\{g(x), g(y)\}$ which implies $g(z) = \alpha g(z) + (1-\alpha)g(z) \geq \alpha g(x) + (1-\alpha)g(y)$. Assume $z < b$. Define $\alpha' \in (0,1)$ by $z = \alpha' x + (1-\alpha')b$, i.e., $\alpha' = (b-z)/(b-x)$. Then $\alpha = (y-z)/(y-x) > \alpha'$ since $y > b$. Hence

$$\begin{aligned} g(z) &= g(\alpha' x + (1-\alpha')b) \\ &\geq \alpha' g(x) + (1-\alpha')g(b) \\ &= \alpha' g(x) + (1-\alpha')g(y) \\ &\geq \alpha g(x) + (1-\alpha)g(y) \end{aligned}$$

where the first inequality follows because f is concave and the second inequality follows because $g(y) \geq g(x)$ implies $\alpha g(x) + (1-\alpha)g(y)$ is a nonincreasing function of α $\square$

Theorem 2.2: If com $V = 1$ then V can be given a complexity 1 representation with all u_i nondecreasing.

Proof: Let $u_i: [0,1] \to R$, $i \in N$, be a representation of V. (Thus the u_i's are continuous and concave.) For $i \in N$ let m_i be such that $u_i(m_i) = \max\{u_i(x) | x \in [0,1]\}$. We define $\{\bar{u}_i: [0,1] \to R | i \in N\}$ as follows. If $\sum_{i \in N} m_i \geq 1$ put

$$\bar{u}_i(x) = \begin{cases} u_i(x) & x \in [0,m_i] \\ u_i(m_i) & x \in (m_i,1], \end{cases} \tag{2.2.1}$$

and

if $\sum_{i \in N} m_i < 1$ put

$$\bar{u}_i(x) = \begin{cases} u_i(m_i) & x \in [0,m_i] \\ u_i(x) & x \in (m_i,1]. \end{cases} \tag{2.2.2}$$

Suppose we have the case $\sum_{i \in N} m_i \geq 1$. Then $\bar{u}_i$ are concave, nondecreasing and continuous by Lemma 2.1. We must show they generate V. Since $\bar{u}_i \geq u_i$, $i \in N$, it follows that $V \subset \{x \in R^n | x_i \leq \bar{u}_i(y_i);\ y_i \geq 0,\ \sum y_i = 1\}$. Conversely, suppose $x \in R^n$, $x_i \leq \bar{u}_i(y_i)$, $y_i \geq 0$, $\sum y_i = 1$. Suppose $y_k > m_k$ for some $k \in N$. Since $\sum_{i \in N} m_i \geq 1$ there exists an $\ell \in N$ such that $y_\ell < m_\ell$. But then y_k can be decreased and y_ℓ increased so that $\sum_{i \in N} y_i = 1$ is maintained, $\bar{u}_k(y_k)$ is unchanged and $\bar{u}_\ell(y_\ell)$ is at worst increased. Thus we may assume that $y_i \leq m_i$ for all $i \in N$. But then $x_i \leq \bar{u}_i(y_i) = u_i(y_i)$, $i \in N$, which implies $x \in V$. This completes the proof if $\sum_{i \in N} m_i \geq 1$. In case $\sum_{i \in N} m_i < 1$ the proof is similar. In particular the $\bar{u}_i$ are nonincreasing, concave and continuous since $\bar{u}_i(1-x)$ is nondecreasing, concave and continuous by Lemma 2.1.

It is a consequence of the above arguments that we may now assume V has a representation with all u_i nonincreasing. In this case we define $\bar{u}_i$, $i \in N$, by

$$\bar{u}_i(x) = \begin{cases} u_i(1 - (n-1)x) & x \in [0,1/(n-1)] \\ u_i(0) & x \in (1/(n-1),1]. \end{cases}$$

The $\bar{u}_i$ are continuous, concave and nondecreasing by Lemma 2.1. Let $\bar{V}$ be the attainable set for $\bar{u}_1,\ldots,\bar{u}_n$. To see that $\bar{V} = V$ take $x_i \geq 0$, $\sum_{i\in N} x_i = 1$ so that $z = (\bar{u}_1(x_1),\ldots,\bar{u}_n(x_n)) \in \bar{V}$. As above it follows that we may assume $x_i \leq 1/(n-1)$, $i \in N$, since each $\bar{u}_i$ is constant for $x_i \geq 1/(n-1)$. Put $y_i = 1 - (n-1)x_i$. Then $y_i \geq 0$, $\sum_{i\in N} y_i = \sum_{i\in N} [1 - (n-1)x_i] = n - (n-1)\sum_{i\in N} x_i = 1$ and $u_i(y_i) = \bar{u}_i(x_i)$ for each i. Thus $z \in V$. Conversely, if $x_i \geq 0$ and $\sum_{i\in N} x_i = 1$ so that $z = (u_1(x_1),\ldots,u_n(x_n)) \in V$, then let $y_i = (1-x_i)/(n-1)$. Then $y_i \geq 0$, $\sum_{i\in N} y_i = 1$ and $\bar{u}_i(y_i) = u_i(x_i)$, so $z \in \bar{V}$. This completes the proof. $\square$

Remark: One cannot in general make the u_i nonincreasing. This follows from the observation that if com $V = 1$ and V has a nonincreasing representation, then there is $z^i \in V$, $i \in N$, where $z^i_j = b_j = \sup\{x_j | x \in V\}$ for $j \neq i$ and z^i_j is "sufficiently small." To prove the observation note that if the u_i are nonincreasing then $u_i(0) = b_i$. For an example consider $V = \text{conv}\{(1,0,0),(0,1,0),(0,0,1)\} - R^3_+$ given by the utilities $u_i(x) = x$, $x \in [0,1]$. There is no $x \in V$ with $x_1 = x_2 = 1$.

Lemma 2.3: Suppose $\lambda_i > 0$ and $c_i \in R$, $i \in N$. Then com $V = m$ if and only if $\text{com}(\{y \in R^n | y_i = \lambda_i x_i + c_i, x \in V\}) = m$.

Proof: If $\{u^i | i \in N\}$ is a complexity m representation for V then $\{\lambda_i u^i + c_i | i \in N\}$ is a complexity m representation for $\{y \in R^n | y_i = \lambda_i x_i + c_i, x \in V\}$. For the converse, take $\lambda_i' = 1/\lambda_i$ and

$c_i' = -c_i/\lambda_i$, $i \in N$. $\square$

Since every attainable set is of the form $C - R^n_+$ where C is compact and convex, it is a consequence of Lemma 2.3 that we may (and will) assume every V satisfies

(2.4) $$b_i = \sup\{x_i \mid x \in V\} = 1 \text{ for each } i \in N, \text{ and}$$

(2.5) $$P \subset R^n_+$$

Lemma 2.6: If com $V = 1$ then there are nondecreasing, nonnegative functions $\{u_i \mid i \in N\}$ which represent V and satisfy $u_i(1) = 1$ for $i \in N$.

Proof: By Theorem 2.2, we may assume we have a representation with all the u_i nondecreasing. Let $m_i = \inf\{x \mid u_i(x) \geq 0\}$. We must have $\sum_{i\in N} m_i < 1$ by (2.4) and (2.5). For if $\sum_{i\in N} m_i > 1$, then whenever $\sum_{i\in N} x_i = 1$, we must have an i such that $x_i < m_i$. From this we conclude that each $y \in P$ has a negative coordinate, contrary to (2.5). If $\sum_{i\in N} m_i = 1$, then we must have $P = \{0\}$ by (2.5), which contradicts (2.4).

For each $i \in N$, define $\bar{u}_i : [0,1] \to R_+$ by

$$\bar{u}_i(x) = u_i([1 - \sum_{j\in N} m_j]x + m_i).$$

The $\bar{u}_i$ are clearly nondecreasing, concave and continuous, as well as non-negative $(\bar{u}_i(0) = u_i(m_i) \geq 0)$. We first show that $\{\bar{u}_i \mid i \in N\}$ represents V. To this end take $y \in I^n$ such that $\sum_{i\in N} y_i = 1$. Put $x_i = (1 - \sum_{j\in N} m_j)y_i + m_i$.

Then $x_i \geq 0$, $\sum_{i\in N} x_i = (1 - \sum_{j\in N} m_j) \sum_{i\in N} y_i + \sum_{i\in N} m_j = 1$ and $\bar{u}_i(y_i) = u_i(x_i)$. Conversely, take $x \in I^n$ such that $\sum_{i\in N} x_i = 1$ and $(u_1(x_1),\ldots,u_n(x_n)) \in P$. Then $u_i(x_i) \geq 0$ by (2.5), and hence $x_i \geq m_i$. Put $y_i = (x_i - m_i)/(1 - \sum_{i\in N} m_j)$. Then $y_i \geq 0$, $\sum_{i\in N} y_i = 1$ and $\bar{u}_i(y_i) = u_i(x_i)$. This proves $\{\bar{u}_i | i \in N\}$ represents V.

To show that $\bar{u}_i(1) = 1$ for each i, note first that (2.4) implies $\bar{u}_i(x) \leq 1$ for $x \in [0,1]$. Since $V = C - R^n_+$, C compact, it follows from (2.4) that, for each i, there exists $x \in V$ such that $x_i = 1$. But the $\bar{u}_i$ generate V, so that there is a $t_i \in [0,1]$ such that $u_i(t_i) \geq 1$. Thus $\bar{u}_i$ nondecreasing implies $\bar{u}_j(1) = 1$. □

<u>Theorem 2.7</u>: Assume V satisfies (2.4), (2.5) and com $V \geq 1$. Then com $V = 1$ if and only if $V = \{x \in I^n | \sum_{i\in N} h_i(x_i) = 1\} - R^n_+$ where, for each i, $h_i = [0,1] \to [0,1]$ is convex, nondecreasing and continuous, and $h_i(0) = 0$. Further $P \subset \{x \in I^n | \sum_{i\in N} h_i(x_i) = 1\}$ and $V \cap I^n = \{x \in I^n | \sum_{i\in N} h_i(x_i) \leq 1\}$.

<u>Proof</u>: Suppose com $V = 1$ and take a representation $\{u_i | i \in N\}$ as specified in Lemma 2.6. Let $m_i = \inf\{x \in [0,1] | u_i(x) = 1\}$. By continuity $u_i(m_i) = 1$. By concavity u_i is strictly increasing on $[0,m_i]$ and hence, by continuity, a bijection of $[0,m_i]$ onto $[u_i(0),1] \subset [0,1]$. Let h_i be the inverse of u_i; $h_i: [u_i(0),1] \to [0,m_i]$. h_i is clearly convex, nondecreasing and continuous. Extend h_i to be defined on $[0,1]$ by $h_i(x) = h_i(u_i(0)) = 0$ for $x \in [0,u_i(0))$. Thus extended $-h_i(1-x)$ is concave, nondecreasing and continuous by Lemma 2.1. We conclude that h_i is convex, nondecreasing and continuous.

To complete this half of the proof it remains to show $V = \{x \in I^n | \sum_{i\in N} h_i(x_i) = 1\} - R^n_+$. For this it suffices to verify

$P \subset \{x \in I^n | \sum_{i\in N} h_i(x_i) = 1\} \subset V$. Take $x \in P$. Then $x = (u_1(y_1),\ldots,u_n(y_n))$ for $y_i \geq 0$, $\sum_{i\in N} y_i = 1$. From com $V \geq 1$ we deduce $y_k < m_k$ for some $k \in N$, and so if $y_i > m_i$ for some $i \in N$ then there exists an $\bar{x} \geq x$ with $\bar{x} \in V$ and $\bar{x}_k > x_k$. This contradiction to $x \in P$ proves $y_i \leq m_i$ for all $i \in N$. Thus $\sum_{i\in N} h_i(x_i) = \sum_{i\in N} h_i(u_i(y_i)) = \sum_{i\in N} y_i = 1$, and so by (2.4) and (2.5) $x \in \{x \in I^n| \sum_{i\in N} h_i(x_i) = 1\}$. Now suppose $x \in \{x \in I^n| \sum_{i\in N} h_i(x_i) = 1\}$. Note that $h_i(x_i) \geq 0$. Further since $u_i h_i(x) = x$ for $x \in [u_i(0),1]$ and $u_i h_i(x) = u_i(0)$ for $x \in [0,u_i(0)]$ it follows that $u_i h_i(x_i) \geq x_i$. Thus $x \in V$.

To prove the second half of the theorem suppose $V = \{x \in I^n| \sum_{i\in N} h_i(x_i) = 1\} - R^n_+$ with the h_i as specified in the theorem. Put $m_i = \sup\{x \in [0,1] | h_i(x) = 0\}$, $i \in N$. As in the first part of the proof h_i is a bijection of $[m_i,1]$ onto $[0,h_i(1)]$. Again, as in the first part of the proof we let u_i be the inverse of h_i and extend u_i to $[0,1]$ by $u_i(x) = 1$ for $x \in [h_i(1),1]$ obtaining a continuous, concave and nondecreasing function. By assumption com $V \geq 1$. To show com $V = 1$ we show $P \subset \{y \in R^n | y_i \leq u_i(x_i),\ x_i \geq 0,\ \sum_{i\in N} x_i = 1\} \subset V$. Take $y \in P$. By the definition of P it follows from $V = \{y \in R^n| \sum_{i\in N} h_i(y_i) = 1\} - R^n_+$ that $\sum h_i(y_i) = 1$. But $h_i(y_i) \geq 0$ by assumption. Further $u_i h_i(\alpha) = \alpha$ for $\alpha \geq m_i$ and for $\alpha < m_i$ $u_i h_i(\alpha) = m_i$ so that $u_i h_i(y_i) \geq y_i$. Conversely, suppose $y \in R^n$, $y_i \leq u_i(x_i)$, $x_i \geq 0$ and $\sum_{i\in N} x_i = 1$. It is easy to see that we may assume $x_i \leq h_i(1)$ for each i. Otherwise V is the corner on $(1,\ldots,1)$ and com $V = 0$. Hence $\sum_{i\in N} h_i u_i(x_i) = \sum_{i\in N} x_i = 1$ and so $y \in V$.

Finally, $V \cap I^n = \{x \in I^n| \sum_{i\in N} h_i(x_i) \leq 1\}$ is a consequence of the h_i's being nondecreasing and continuous. □

Example: It follows from Theorem 2.7 that for an integer $n \geq 2$, com $(S^{n-1}-R^n_+) = 1$ with $h_i(t) = t^2$, where S^{n-1} is the $n-1$ sphere in R^n. A set $\{u_i \mid i \in N\}$ which represents $S^{n-1} - R^n_+$ is given by $u_i(t) = \sqrt{t}$.

§3. SOME FURTHER PROPERTIES

We assume (2.4) and (2.5) throughout this section. Unless otherwise stated we will also use the convention that h_i denotes a convex, nondecreasing, continuous function from $[0,1]$ to $[0,1]$ which satisfies $h_i(0) = 0$.

Theorem 3.1: com $V \leq 1$ implies P is closed.

Proof: If com $V = 0$, P is a single point, and therefore closed.

Suppose com $V = 1$. We have $P \subset \{x \in I^n \mid \sum_{i \in N} h_i(x_i) = 1\} \subset V$ by Theorem 2.7. To show P is closed take $\{x^j\} \subset P$ with $x^j \to x$. We show $x \in P$. By continuity $\sum_{i \in N} h_i(x_i) = 1$. If $x \notin P$ there exists $\bar{x} \in P$ such that $\bar{x} \geq x$ and $\bar{x}_k > x_k$ for some $k \in N$. Since each h_i is increasing this implies $h_k(\bar{x}_k) = h_k(x_k)$ and hence $h_k(x) = h_k(\bar{x}_k)$ for $x \in [0,\bar{x}_k]$. But $x^j_k \to x_k$ implies $x^j_k < \bar{x}_k$ for sufficiently large j. For such a large j let $\bar{x}^j_i = x^j_i$ for $i \neq k$ and $\bar{x}^j_k = \bar{x}_k$. Then $\bar{x}^j \in V$, $\bar{x}^j \geq x^j$ and $\bar{x}^j_k > x^j_k$ contradicting $x^j \in P$. We conclude $x \in P$. $\square$

Example: Let

$$A = \{x \in R^3_+ \mid x_1^2 + x_3^2 = 1,\ x_2 = 0\},$$

$$B = \text{conv } (A \cup \{(0,1,1)\}) \quad \text{and}$$

$$V = B - R^3_+.$$

Then $P = B \backslash \{x \in R^3_+ | x_1 = 0,\ x_3 = 1$ and $x_2 < 1\}$ which is not closed (see [1]). Thus $\mathrm{com}\ V \geq 2$. To see that $\mathrm{com}\ V = 2$ define $\{u_i = I^2 \to R | i \in N\}$ by $u_2(x,y) = x$, $u_3(x,y) = y$ and

$$u_1(x,y) = \sup\{z | (z, 1-x, 1-y) \in V\}.$$

u_1 is concave and continuous since V is convex and closed, and u_1 is defined over a polyhedral set (namely I^2).

Define a_i and b_i, $i \in N$, by

$$a_i = \inf\{x_i | x \in P\} \text{ and}$$

$$b_i = \sup\{x_i | x \in P\} = \sup\{x_i | x \in V\}.$$

By (2.4), we have that $b_i = 1$ for each i.

<u>Theorem 3.2</u>: Suppose that V satisfies $a_i = 0$ if $a_i < 1$, and $\mathrm{com}\ V \neq 0$. Then $\mathrm{com}\ V = 1$ if and only if $V = \{x \in I^n | \sum_{i \in N} h_i(x_i) = 1\} - R^n_+$ where the h_i are strictly increasing when $a_i = 0$ and constant when $a_i = 1$.

<u>Proof</u>: If $V = \{x \in I^n | \sum_{i \in N} h_i(x_i) = 1\} - R^n_+$ then by Theorem 2.7, $\mathrm{com}\ V = 1$.

Suppose $\mathrm{com}\ V = 1$. By Theorem 2.7 we have $V = \{x \in I^n | \sum_{i \in N} h_i(x_i) = 1\} - R^n_+$, where the h_i are as specified at the beginning of this section. Suppose $k \in N$ and $a_k = 0$. Then by Theorem 3.1 there is an $x \in P$ such that $x_k = 0$. By Theorem 2.7 $\sum_{i \in N} h_i(x_i) = 1$. If h_k is not strictly increasing then by convexity since h_k nondecreasing there is a $t > 0$ such that $h_k(t) = 0$. Define $\bar{x} \in R^n$ by $\bar{x}_i = x_i$ for $i \neq k$ and $\bar{x}_k = t$. Then

$\sum h_i(\bar{x}_i) = 1$ which implies $\bar{x} \epsilon V$ contradicting $x \epsilon P$. Hence, h_k is strictly increasing. Assume $a_k = 1$. Let $m_i = \sup\{t | h_i(t) = 0\}$ for each i. Since $h_k(0) = 0$ we wish to show $m_k = 1$ (i.e., $m_i = a_i$ for each i). We claim that $A = \{x \epsilon I^n | x_i \geq m_i, \sum_{i \epsilon N} h_i(x_i) = 1\} = P$. Clearly $P \subset A$. $A \subset P$ follows because every h_i is convex and nondecreasing and so $t_1 > t_2 \geq m_i$ implies $h_i(t_1) > h_i(t_2)$. This proves the claim. Suppose $m_k < 1$ and let $x \epsilon I^n$ be defined by $x_i = 1$ for $i \neq k$ and $x_k = m_k$. If $\sum_{i \epsilon N} h_i(x_i) < 1$, we note that $\sum_{i \epsilon N} h_i(1) > 1$ since com $V \neq 0$ so that by continuity there is an $\bar{x} \geq x$ with $1 > \bar{x}_k > x_k$ such that $\sum_{i \epsilon N} h_i(\bar{x}_i) = 1$. By the claim, we have $\bar{x} \epsilon P$, contradicting $a_k = 1$. If $\sum_{i \epsilon N} h_i(x_i) \geq 1$ we note that $\sum_{i \epsilon N} h_i(m_i) = 0$ so that by continuity there is an $\bar{x} \leq x$ with $\bar{x}_i \geq m_i$ for each i, $\bar{x}_k = m_k$ and $\sum_{i \epsilon N} h_i(\bar{x}_i) = 1$. The claim then implies $\bar{x} \epsilon P$ contradicting $a_k = 1$. We have proved that the assumption $m_k < 1$ is untenable, and thus that $h_k \equiv 0$. □

We now observe some corollaries of Theorem 3.2 and its proof.

Corollary 3.2.1: Suppose $V = \{x \epsilon I^n | \sum_{i \epsilon N} h_i(x_i) = 1\} - R^n_+$. Then if $a_i = 0$, h_i is strictly increasing and if $a_i = 1$, $h_i \equiv 0$. □

Corollary 3.2.2: Suppose $V = \{x \epsilon I^n | \sum_{i \epsilon N} h_i(x_i) = 1\} - R^n_+$. Then $P = \{x \epsilon I^n | x_i \geq a_i, \sum_{i \epsilon N} h_i(x_i) = 1\}$ and thus if $a_i = 0$ for each i, $P = \{x \epsilon I^n | \sum_{i \epsilon N} h_i(x_i) = 1\}$. □

One can see either directly or from Corollary 3.2.1 that components i for which $a_i = b_i$ do not affect complexity. Thus, in future results we will make the assumption that $a_i = 0$ for each $i \epsilon N$. By Corollary 3.2.1 this assumption implies that the h_i are strictly increasing.

Before giving the next result we introduce some necessary notation. If $x \in R^n$ and $S \subset N$ we let $x^{(S)} \in R^{|S|}$ denote the vector with components x_i for $i \in S$ (i.e., the projection of x onto $R^{|S|}$). Given V and $t \in I^{n-|S|}$ define $V^S(t) = \{x^{(S)} | x \in V, x^{(N\setminus S)} = t\}$. For $\{i,j\} \subset N$ we use the conventions $V^{i,j}(t) = V^{\{i,j\}}(t)$ and $V^i(t) = V^{\{i\}}(t)$. By Theorem 2.3 of [3], $V^S(t)$ is an attainable set so that we are justified in taking its Pareto surface $P^S(t)$.

Theorem 3.3: If $\text{com } V = 1$, $a_i = 0$ for all $i \in N$, $S \subset N$, and $t^1 \neq t^2 \in I^{n-|S|}$ with $t_1 \leqq t_2$, then $x \in P^S(t^1) \cap V^S(t^2)$ implies $x = e^{|S|}$. If $|S| = n - 1$ then $P^S(t^1) \cap V^S(t^2) = \emptyset$.

Proof: Suppose $x \in P^S(t^1) \cap V^S(t^2)$ (this implies $S \neq \emptyset$). From (2.5), it follows that $x \geqq 0$. By Theorems 2.7 and 3.2, $V \cap I^n = \{x \in I^n | \sum_{i \in N} h_i(x_i) \leqq 1\}$ where the h_i are strictly increasing.

Take $y, z \in V$ such that $y^{(N\setminus S)} = t^1$, $z^{(N\setminus S)} = t^2$ and $y^{(S)} = z^{(S)} = x$. Since $t^1 \neq t^2$ there is a $k \notin S$ such that $t^1_k < t^2_k$. But h_k is strictly increasing so $h_k(y_k) < h_k(z_k)$. We conclude that $\sum_{i \in N} h_i(y_i) < \sum_{i \in N} h_i(z_i) \leqq 1$. By the continuity of h_i, $i \in N$, we can increase y_i if $y_i < 1$ and maintain $\sum_{i \in N} h_i(y_i) < 1$. Since this contradicts $x \in P^S(t_1)$, we conclude that $x = e^{|S|}$. To complete the proof suppose $S = N\setminus\{k\}$. By Theorem 3.1 there is a $w \in P$ such that $w_k = a_k = 0$. Since $y \geqq w$ we deduce $y \in P$. But $z \geqq y$ with $z_k = t^2 > t^1 = y_k$ which implies $y \notin P$ and so x cannot exist. $\square$

The following example was discussed in [3].

Example: Consider $V \subset R^3$ given by $P - R^3_+$ where P is the line joining the points $(1,0,0)$ and $(0,1,1)$. Clearly $\text{com } V \geqq 1$. But $(0,1) \in P^{1,2}(0) \cap V^{1,2}(1)$ so $\text{com } V \geqq 2$ by Theorem 3.3. A representation over I^2 is given in [3], which shows $\text{com } V = 2$.

For $S \subset N$ and $t \in I^{n-|S|}$ define $\bar{P}^S(t) = \{x^{(S)} | x \in P, x^{(N\setminus S)} = t\}$. With this notation we have the following result which says essentially that the operation of computing the Pareto surface commutes with that of projection.

Corollary 3.3.1: If $\text{com } V = 1$, $a_i = 0$ for all $i \in N$, $S \subset N$, $t \in I^{n-|S|}$ and $V^S(t) \neq \emptyset$, then $\bar{P}^S(t) \neq \emptyset$ implies $\bar{P}^S(t) = P^S(t)$ and $\bar{P}^S(t) = \emptyset$ implies $P^S(t) = e^{|S|}$.

Proof: Clearly $\bar{P}^S(t) \subset P^S(t)$. If $\bar{P}^S(t) \neq \emptyset$ then take $x \in P^S(t)$ and $y \in V$ such that $(y^{(S)}, y^{(N\setminus S)}) = (x,t)$. If $y \in P$ we are done. Else there is a $z \in P$ such that $z \geqq y$, $z \neq y$. If $z^{(N\setminus S)} = y^{(N\setminus S)} = t$ then $z^{(S)} \geqq x$, $z^{(S)} \neq x$ implies $x \notin P^S(t)$. Thus $z^{(N\setminus S)} \geqq t$ but $z^{(N\setminus S)} \neq t$. This implies $x \in V^S(z^{(N\setminus S)})$ since $x \leqq z^{(S)}$, and we deduce from Theorem 3.3 that $x = e^{|S|}$. But then $\bar{P}^S(t) \neq \emptyset$ implies $x \in \bar{P}^S(t)$ and so $\bar{P}^S(t) = P^S(t)$.

Suppose $\bar{P}^S(t) = \emptyset$ and take $x \in P^S(t) \neq \emptyset$ and $y \in V$ such that $(y^{(S)}, y^{(N\setminus S)}) = (x,t)$. Since $\bar{P}^S(t) = \emptyset$, $y \notin P$. Take $z \in P$ such that $z \geqq y$, $z \neq y$. Again $\bar{P}^S(t) = \emptyset$ implies $z^{(N\setminus S)} \neq y^{(N\setminus S)} = t$. Thus by Theorem 3.3, $x = e^{|S|}$. $\square$

Let us define a proper face of V to be a set of the form $V^S(0)$ where $S \subsetneq N$ (we assume $a_i = 0$, $i \in N$). A conjecture that seems natural is the following. If $\text{com } V \leqq 1$ then V is uniquely determined by its proper faces. This conjecture is clearly true for $\text{com } V = 0$ (and false for com $V \geq 2$). The following example shows it is false for com $V = 1$.

Example: Let $V = S^2 - R^3_+$. As we have already seen, a representation for V is given by $h_i(t) = t^2$ for each i. Now for each $i \neq j$, $V^{ij}(0)$ is essentially $S^1 - R^2_+$. Define $\hat{h}_i : I \to I$ by

$$\hat{h}_i(t) = \begin{cases} t/\sqrt{2} & t \leq 1/\sqrt{2} \\ 1 - \sqrt{(1-t^2)/2} & t \geq 1/\sqrt{2}, \end{cases}$$

and let

$$\hat{V} = \{x \in I^3 | \sum \hat{h}_i(x_i) = 1\} - R^3_+.$$

Since the $\hat{h}_i$ can be seen to be convex, continuous and increasing, $\hat{V}$ is a complexity 1 attainable set. Now

$$\hat{V}^{jk}(0) = \{x \in I^2 | \hat{h}_j(x_j) + \hat{h}_k(x_k) = 1\} - R^2_+.$$

Note that $(x_j, x_k) \in \hat{V}^{jk}(0)$ implies that one coordinate is in $[0, 1/\sqrt{2}]$ while the other is in $[1/\sqrt{2}, 1]$. Suppose $x_j \leq 1/\sqrt{2}$. Thus

$$x_j/\sqrt{2} + 1 - \sqrt{(1-x_k^2)/2} = 1$$

if and only if

$$x_j/\sqrt{2} = \sqrt{(1-x_k^2)/2}$$

if and only if

$$\frac{1}{2}x_j^2 = \frac{1}{2}(1-x_k)^2,$$

that is, $x_j^2 + x_k^2 = 1$.

Hence, $\hat{V}^{jk}(0) = V^{jk}(0)$ for each $j \neq k$. But $(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3}) \notin \hat{V}$ since $\sqrt{3}/\sqrt{2} > 1$. Thus we have two different complexity 1 attainable sets with identical proper faces, since clearly $V^i(0) = \hat{V}^i(0) = (-\infty,1]$ for each i.

Although, as illustrated above, com $V = 1$ and $\{V^S(0) | S \neq N\}$ do not determine V in general, there are some special conditions under which they do determine V. We begin with a lemma.

Lemma 3.4: Suppose $V = \{x \in I^n | \sum_{i\in N} h_i(x_i) = 1\} - R^n_+$ and $a_i = 0$ for $i \in N$. Then if h_k is surjective for some $k \in N$, and $i \in S$, where $k \in S \subset N$, then h_i is determined by h_k and $V^S(0)$.

Proof: Since $V^{i,k}(0) = (V^S(0))^{i,k}(0)$, we may assume $S = \{i,k\}$. Take $x \in [0,1]$. Since h_k is surjective, continuous and strictly increasing there will be for a given value $h_i(x)$ a unique $y \in [0,1]$ such that $h_i(x) + h_k(y) = 1$. Let $z \in I^n$ have $z^{\{i,k\}} = (x,y)$ and $z^{(N\setminus\{i,k\})} = 0$. Then $\sum_{i\in N} h_i(z_i) = 1$ and so $z \in P$ which implies, by Corollary 3.3.1, $z^{(\{i,k\})} \in P^{i,k}(0)$. But for a given x there can be at most one y such that $(x,y) \in P^{i,k}(0)$. Thus x uniquely determines y which in turn uniquely determines $h_i(x)$: $h_i(x) \equiv 1 - h_k(y)$. $\square$

Let $T^{n-1} = \{x \in R^n_+ | \sum_{i\in N} x_i = 1\}$ be the $n - 1$ simplex in R^n.

Lemma 3.5: Suppose $V = \{x \in I^n | \sum_{i\in N} h_i(x_i) = 1\} - R^n_+$, and $a_i = 0$ for $i \in N$. If $S \subset N$ with $|S| \geq 2$ and $V^S(0) = T^{|S|-1}$ then $h_i(x) = x$ for $i \in S$.

Proof: We may assume $S = \{i,j\}$. Suppose $x \in I^n$ with $x^{(N\setminus S)} = 0$,

$x_j \in (0,1)$ and $x_j = 1 - x_i$. Then $x^{(S)} \in P^S(0)$ and so by Corollary 3.3.1 $x \in P$. Since $\sum_{k \in N} h_k(x_k) = 1$ and $h_k(0) = 0$ for each k we have $h_i(x_i) + h_j(1-x_i) = 1$. But then $h_i(x_i) = 1 - h_j(1-x_i)$ is concave and convex which implies h_i is affine and $h_i(1) = 1 - h_j(0) = 1$. Thus $h_i(0) = 0$ implies $h_i(x) = x$. Similarly $h_j(x) = x$. $\square$

<u>Theorem 3.6</u>: Suppose $V = \{x \in I^n | \sum_{i \in N} h_i(x_i) = 1\} - R^n_+$, and $a_i = 0$ for $i \in N$. If $k \in S \subseteq N$ with $|S| \geq 2$ and $V^S(0) = T^{|S|-1}$, then V is determined by $\{V^L(0) \mid |L| = 2, k \in L\}$.

<u>Proof</u>: By Lemma 3.5, h_k is determined by $V^L(0)$ for some $k \in L \subset S$ with $|L| = 2$. In fact, $h_k(x) = x$ and so h_k is surjective. Then by Lemma 3.4 the h_i for $i \neq k$ are determined by $\{V^L(0) \mid |L| = 2, k \in L\}$. $\square$

It is not hard to see that we can replace $\{V^L(0) \mid |L| = 2, k \in L\}$ in Theorem 3.6 by $\{V^L(0) | L \in B\}$ where $B \subset \{L | L \subset N\}$ is such that $L \subset S$ for some $L \in B$, and for each $i \in N$ there is an $L \in B$ such that $\{i,k\} \subset L$.

We conclude with a rather curious property of complexity 1 attainable sets.

<u>Theorem 3.7</u>: Let $V \subset R^n$ be an attainable set such that $\text{com}\, V = 1$ and $a_i = 0$ for each i. Suppose $i_1, \ldots, i_k$ are not necessarily distinct elements of N where $k > 1$ is odd and $i_1 = i_k$. Suppose also that $x_1, \ldots, x_k$ are such that for $1 \leq j < k$, $(x_j, x_{j+1}) \in \overline{P}^{i_j i_{j+1}}(0)$. Then $x_1 = x_k$.

<u>Proof</u>: By Theorem 3.2, we have a representation

$$V = \{x \in I^n | \sum_{i \in N} h_i(x_i) = 1\} - R^n_+$$

with the h_i strictly increasing, and by Corollary 3.2.2,

$$P = \{x \in I^n | \sum_{i \in N} h_i(x_i) = 1\}.$$

Now for each j, $1 \leq j < k$,

$$\bar{P}^{i_j i_{j+1}}(0) = \{x \in I^2 | h_{i_j}(x_1) + h_{i_{j+1}}(x_2) = 1\}.$$

The hypothesis thus implies that $h_{i_j}(x_j) = h_{i_{j+2}}(x_{j+2})$ for $1 \leq j \leq k - 2$, and hence $h_{i_1}(x_1) = h_{i_k}(x_k)$. Since $i_1 = i_k$ and h_{i_1} is strictly increasing $x_1 = x_k$. $\square$

<u>Example</u>: Let $V \subset R^3$ be given by

$$V = \text{conv } (A^{23} \cup A^{13} \cup A^{12})$$

where $$A^{23} = \{x \in R_+^{\{2,3\}} | x_2 + x_3 = 1\},$$

$$A^{13} = \{x \in R_+^{\{1,3\}} | x_1 + x_3 = 1\}, \text{ and}$$

$$A^{12} = \{x \in R_+^{\{1,2\}} | x_1 = 1 + x_2(1-\sqrt{2}),\ x_1 \geq 1/\sqrt{2}\}$$

$$\cup \{x \in R_+^{\{1,2\}} | x_1^2 + x_2^2 = 1,\ x_1 \leq 1/\sqrt{2}\}.$$

Now the sets $\bar{P}^{ij}(0)$ are essentially given by the sets A^{ij}. Let $i_1,\ldots,i_7$ be the sequence 3,1,2,3,1,2,3. Let $(x_1,\ldots,x_7) = (1/2, 1/2, \sqrt{3}/2, 1-\sqrt{3}/2, \sqrt{3}/2, (\sqrt{3}/2-1)/(1-\sqrt{2}), (2-\sqrt{2}-\sqrt{3}/2)/(1-\sqrt{2}))$. It can be checked that $(x_1,\ldots,x_7)$

satisfies the requirements of the theorem, yet $x_1 \neq x_7$. We conclude that com $V \neq 1$. We could have obtained this conclusion from Lemma 3.5 as well.

We remark that Theorem 3.7 provides a necessary and sufficient condition for the existence of h_i's, not necessarily convex, which give $\{P^S(0) | S \subset N, |S| = 2\}$.

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